

Boundary S-matrix for the integrable q-Potts model

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1995 J. Phys. A: Math. Gen. 28 7039

(<http://iopscience.iop.org/0305-4470/28/24/005>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.68

The article was downloaded on 02/06/2010 at 00:57

Please note that [terms and conditions apply](#).

Boundary S -matrix for the integrable q -Potts model

Leung Chim†

Department of Physics and Astronomy, Rutgers University, PO Box 849, Piscataway, NJ 08855-0849, USA

Received 24 January 1995

Abstract. The 2D off-critical q -state Potts model with boundaries was studied as a factorizable relativistic scattering theory. The scattering S -matrices for particles reflecting off the boundaries were obtained for the cases of ‘fixed’ and ‘free’ boundary conditions. In the Ising limit, the computed results agreed with recent work [5].

1. Introduction

In the past decade, many advances have been made in two-dimensional statistical mechanical systems by applying the idea of conformal invariance [1]. Their critical points are well described by conformal field theory (CFT) and their universality can be classified by a Virasoro central charge c [2]. When such a model is perturbed off their critical point, the conformal symmetry is broken and the theory develops finite correlation lengths. However, for certain perturbations, residue symmetries survive in the form of an infinite set of commuting integrals of motion and render the theory integrable [3]. Such ‘perturbed conformal field theories’ can often be described by a relativistic scattering theory of massive particles where the S -matrix is factorizable. All physical information about the field theory can then be obtained from the S -matrix by constructing correlation functions using the form-factors method [4].

Recently, statistical systems with boundaries were studied using the above method [5]. It was found that one can choose certain boundary conditions which preserves the integrability of the bulk theory. Such integrable boundary conditions can be represented by the boundary S -matrix describing the scattering of particles from the boundary. In this work, such boundary S -matrices are obtained for the q -state Potts model (for $0 < q < 4$) with free and fixed boundary conditions.

2. Integrable q -states Potts model

In the lattice q -Potts model [6], the spins $a(x)$ at the sites of the lattice are allowed to be in one of the q possible states $(1, 2, \dots, q)$. The partition function has the form

$$Z = \sum_{\{a(x)\}} \prod_{(x,y)=\langle nn \rangle} (1 + K \delta_{a(x),a(y)}) \quad (1)$$

† E-mail: CHIM@ruhets.rutgers.edu

where the product is taken over the nearest neighbours of the lattice, and is invariant under the permutation group S_q . The phase transition point of this model occurs at

$$K = K_c = \sqrt{q}. \quad (2)$$

For $0 < q < 4$, this is a second-order phase transition, and its critical point is described by the CFT with central charge [7]

$$c = 1 - \frac{6}{p(p+1)} \quad \text{where} \quad \sqrt{q} = 2 \sin\left(\frac{\pi}{2} \frac{p-1}{p+1}\right). \quad (3)$$

The energy density $\epsilon(x)$ of the Potts model then corresponds to the degenerate primary field $\Phi_{(2,1)}$ (in the notations of [2]) with dimension

$$\Delta_\epsilon = \frac{1}{4} + \frac{3}{4p}. \quad (4)$$

This field is also a relevant operator in the theory, and perturbation of the CFT action by this field leads to an off-critical theory with action

$$A_{q,\tau} = A_{\text{CFT}(c)} + \tau \int \epsilon(x) d^2x \quad (5)$$

where

$$\tau = \frac{K_c - K}{K_c} \quad (6)$$

and $A_{\text{CFT}(c)}$ is the 'action' of the CFT with central charge c . This action describes the scaling domain of the q -Potts model, and is shown in [8] to be an integrable field theory (i.e. possess non-trivial higher-spin local integrals of motion). Following [8], we consider the low temperature phase $K > K_c$ of this theory. The field theory (5) has q degenerate vacua $|a\rangle$; $a = 1, \dots, q$, with S_q acting by permutations of these vacuum states. Its particle content must then contain $q(q-1)$ 'kinks' A_{ab} ; $a, b = 1, \dots, q$; $a \neq b$, which corresponds to the domain wall separating vacua a and b . The antiparticles can be identified by $\bar{A}_{ab} = A_{ba}$. The masses of these particles are taken to be equal to $M \sim |\tau|^{1/(1-\Delta_\epsilon)}$.

The scattering of these asymptotic particles is governed by the S -matrix and the integrability of (5) implies that this S -matrix is factorizable. Recall that the energy-momentum of particles can be parametrized by their rapidity θ , where

$$E = Mch\theta \quad P = Msh\theta. \quad (7)$$

The asymptotic particles states are generated by the 'particle creation operator' $A_{ab}(\theta)$ satisfying the quadratic commutation relations

$$A_{ab}(\theta_1)A_{bc}(\theta_2) = \sum_{d \neq a, d \neq c} -S_{ac}^{bd}(\theta_{12})A_{ad}(\theta_2)A_{dc}(\theta_1) \quad (8)$$

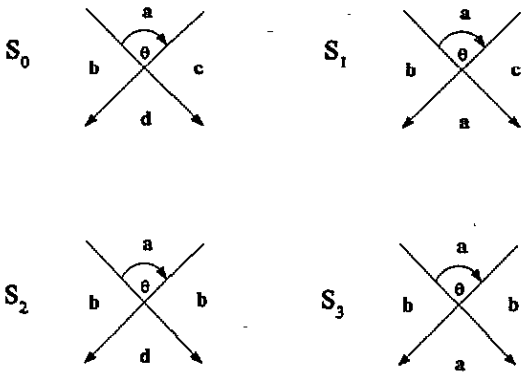


Figure 1. The scattering processes for the bulk S-matrix elements $S_0(\theta)$, $S_1(\theta)$, $S_2(\theta)$ and $S_3(\theta)$ defined in (9); with $a \neq b \neq c \neq d$.

where $\theta_{12} = \theta_1 - \theta_2$. The S_q symmetric two-kink S-matrix elements (figure 1) were computed in [8] to be (for $a \neq b \neq c \neq d$)

$$S_{ac}^{bd}(\theta) = S_0(\theta) = \frac{\text{sh}(\lambda\theta)\text{sh}(\lambda(i\pi - \theta))}{\text{sh}(\lambda(\theta - \frac{2\pi i}{3}))\text{sh}(\lambda(\frac{i\pi}{3} - \theta))} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \tag{9a}$$

$$S_{ac}^{bb}(\theta) = S_1(\theta) = \frac{\sin(\lambda\frac{2\pi}{3}) \text{sh}(\lambda(i\pi - \theta))}{\sin(\lambda\frac{\pi}{3}) \text{sh}(\lambda(\frac{2\pi i}{3} - \theta))} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \tag{9b}$$

$$S_{aa}^{bd}(\theta) = S_2(\theta) = \frac{\sin(\lambda\frac{2\pi}{3}) \text{sh}(\lambda\theta)}{\sin(\lambda\frac{\pi}{3}) \text{sh}(\lambda(\theta - \frac{i\pi}{3}))} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \tag{9c}$$

$$S_{aa}^{bb}(\theta) = S_3(\theta) = \frac{\sin(\lambda\pi)}{\sin(\lambda\frac{\pi}{3})} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \tag{9d}$$

where

$$\lambda = \frac{3p-1}{2p+1} \tag{10}$$

and

$$\Pi(x) = -\frac{\Gamma(1-x)\Gamma(1-\lambda+x)\Gamma(\frac{7}{3}\lambda-x)\Gamma(\frac{4}{3}\lambda+x)}{\Gamma(1+x)\Gamma(1+\lambda-x)\Gamma(\frac{1}{3}\lambda+x)\Gamma(\frac{4}{3}\lambda-x)} \prod_{k=1}^{\infty} \Pi_k(x)\Pi_k(\lambda-x) \tag{11}$$

$$\Pi_k(x) = \frac{\Gamma(1+2k\lambda-x)\Gamma(2k\lambda-x)\Gamma(1+(2k-\frac{1}{3})\lambda-x)\Gamma((2k+\frac{7}{3})\lambda-x)}{\Gamma(1+(2k+1)\lambda-x)\Gamma((2k+1)\lambda-x)\Gamma(1+(2k-\frac{4}{3})\lambda-x)\Gamma((2k+\frac{4}{3})\lambda-x)}$$

The amplitudes $S_1(\theta)$ possess a ‘bound-state’ pole at $\theta = \frac{2\pi i}{3}$ (figure 2(a)) while $S_2(\theta)$ have the ‘cross-channel’ pole at $\theta = \frac{\pi i}{3}$ (figure 2(b)). Likewise $S_0(\theta)$ have both the above poles (figure 2(c)) with residues

$$\text{Res}_{\theta=\frac{2\pi i}{3}} S_0(\theta) = \text{Res}_{\theta=\frac{2\pi i}{3}} S_1(\theta) = -\text{Res}_{\theta=\frac{i\pi}{3}} S_0(\theta) = -\text{Res}_{\theta=\frac{i\pi}{3}} S_2(\theta) = if^2(\lambda) \tag{12}$$

where

$$f(\lambda) = \sqrt{\frac{1}{\lambda} \sin\left(\frac{2\pi}{3}\lambda\right)} \exp\left\{ \int_0^{\infty} \frac{(1-e^{-(1-\frac{4}{3}\lambda)t})(1-e^{-\frac{2}{3}\lambda t})(1-e^{\frac{1}{3}\lambda t})e^{-\lambda t}}{2(1-e^{-t})(1+e^{-\lambda t})} \frac{dt}{t} \right\} \tag{13}$$

is the ‘three-kink coupling’ (figure 3).

As is shown in [8], the complete particle spectrum of field theory (5) for $3 < q < 4$ is quite complicated with the appearance of exotic excitation and bound states. For simplicity, we will restrict our attention to the range $0 < q \leq 3$.

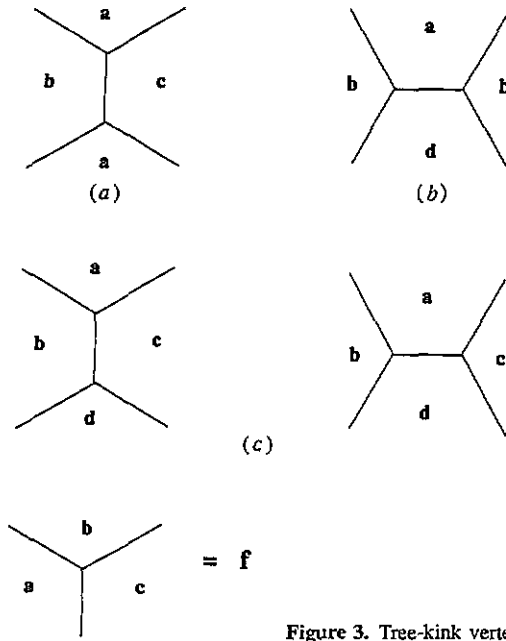


Figure 2. (a) Spacetime diagrams associated with the pole at $\theta = \frac{2\pi i}{3}$ in $S_1(\theta)$ and (b) the corresponding cross-channel pole in $S_2(\theta)$. The amplitude $S_0(\theta)$ possesses both poles as shown in (c); the states a, b, c, d are all different.

Figure 3. Tree-kink vertex associated with the ‘coupling constant’ f .

3. Boundary S -matrix

It is natural to consider the Potts model in the presence of boundaries with some boundary conditions imposed on the boundary spins. In [5], it was shown that certain boundary conditions preserve the integrability of the bulk theory (i.e. an infinite subset of the bulk integrals of motion survives with the introduction of the boundaries). For a relativistic scattering theory of massive particles, one can associate these integrable boundary conditions with certain boundary S -matrices which describe the scattering of particles with the boundary.

To be more precise, consider the model defined on an semi-infinite plane (say $x \in (-\infty, 0], y \in (-\infty, \infty)$, the y -axis being the boundary). Let us suppose that there exist integrable boundary conditions for the Potts model with the modified action

$$A = A_{q,\tau+\text{CBC}} + \int_{-\infty}^{\infty} dy \Phi_B(y) \tag{14}$$

where $A_{q,\tau+\text{CBC}}$ is the action (5) with certain conformal boundary conditions (CBC), and $\Phi_B(y)$ is some relevant boundary operator [9, 5]. One can think of (14) as a perturbation of CBC, and the corresponding Fock states can be classified as asymptotic scattering states. In particular, the boundary with boundary spins in the state ‘ a ’ can be associated with a stationary impenetrable particle B_a of infinite mass at $x = 0$. Then the asymptotic n -kink scattering state can be written as the product

$$A_{a_1 a_2}(\theta_1) A_{a_2 a_3}(\theta_2) \cdots A_{a_{n-1} a_n}(\theta_{n-1}) A_{a_n a}(\theta_n) B_a \tag{15}$$

where the vacua a_1, a_2, \dots, a_n satisfy the restrictions $a_i \neq a_{i+1}$, and $a_n \neq a$.

If the initial ‘in-state’ of scattering is the asymptotic state (15) with $\theta_1 > \theta_2 > \dots > \theta_n > 0$ (i.e. n kinks moving towards the boundary of state ‘ a ’), then in the infinite future,

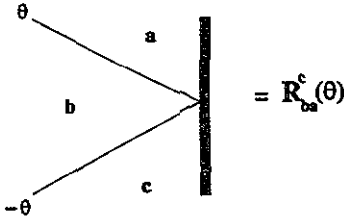


Figure 4. The boundary scattering processes described by the amplitude $R_{ba}^c(\theta)$, with $b \neq a, c$.

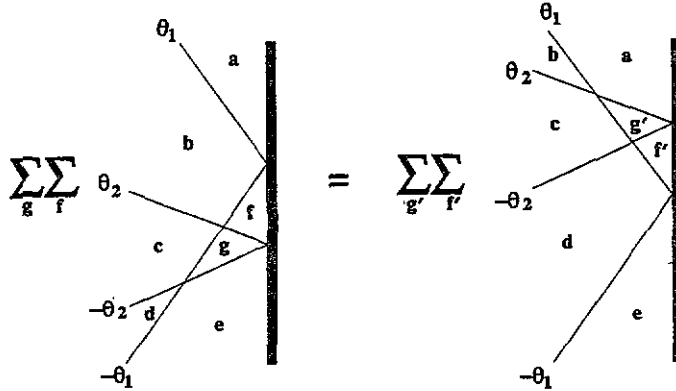


Figure 5. Boundary Yang–Baxter equation. The variables $g, f, g', f' = 1, 2, \dots, q$ satisfy the ‘admissibility conditions’ $g \neq c, e; f \neq b, g; g' \neq a, c; f' \neq d, g'$. Boundary conditions can also place further constraints on these variables.

this state becomes a superposition of the final ‘out-states’. Integrability of (14) constrains ‘out-states’ to have the form

$$A_{b_1 b_2}(-\theta_1) \dots A_{b_{n-1} b_n}(-\theta_{n-1}) A_{b_n b}(-\theta_n) B_b \tag{16}$$

with $b_i \neq b_{i+1}$ and $b_n \neq b$. Thus we have the relation

$$\begin{aligned} & A_{a_1 a_2}(\theta_1) A_{a_2 a_3}(\theta_2) \dots A_{a_{n-1} a_n}(\theta_{n-1}) A_{a_n a}(\theta_n) B_a \\ &= \sum_{b_1} \dots \sum_{b_n} \sum_b R_{a_1 \dots a_n a}^{b_1 \dots b_n b}(\theta_1, \dots, \theta_n) A_{b_1 b_2}(-\theta_1) \dots A_{b_{n-1} b_n}(-\theta_{n-1}) A_{b_n b}(-\theta_n) B_b \end{aligned} \tag{17}$$

which defines the n -kink boundary S -matrix. When $n = 1$, we have the simple commutation relation

$$A_{ba}(\theta) B_a = \sum_c R_{ba}^c(\theta) A_{bc}(-\theta) B_c \tag{18}$$

where $R_{ba}^c(\theta)$ are elements of the boundary S -matrix for the reflection of one particle off the boundary (figure 4). It follows from the factorizability of the scattering that $R_{a_1 \dots a_n a}^{b_1 \dots b_n b}$ can be written as a product of bulk amplitudes S_{ac}^{bd} , and boundary amplitudes R_{ba}^c . For example, when two kinks scatter off the boundary, the amplitude for this scattering can be factorized in two equivalent ways (figure 5), leading to

$$\begin{aligned} & \sum_g \sum_f R_{ba}^f(\theta_1) S_{cf}^{bg}(\theta_1 + \theta_2) R_{gf}^e(\theta_2) S_{ce}^{gd}(\theta_2 - \theta_1) \\ &= \sum_{g'} \sum_{f'} S_{ca}^{bg'}(\theta_2 - \theta_1) R_{g'a}^{f'}(\theta_2) S_{cf'}^{g'd}(\theta_1 + \theta_2) R_{d f'}^e(\theta_1) \end{aligned} \tag{19}$$

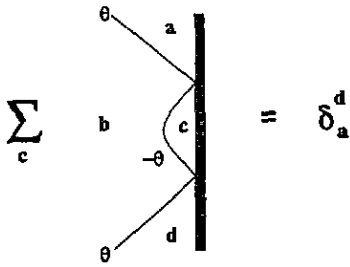


Figure 6. Scattering processes described by the boundary unitarity condition (20); $b \neq a, c$ and $c \neq b$.

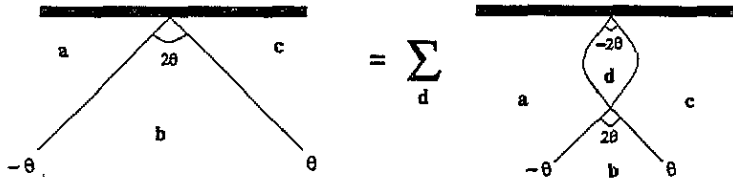


Figure 7. Scattering processes for the cross-unitarity condition (22); $b \neq a, c$ and $d \neq a, c$.

which is known as the ‘boundary Yang–Baxter equation’ [10].

As is known, the amplitudes $R_{ba}^c(\theta)$ have to satisfy several conditions in addition to (19). Firstly we have the ‘boundary unitarity condition’ (figure 6)

$$\sum_{c \neq b} R_{ba}^c(\theta) R_{bc}^d(-\theta) = \delta_a^d \tag{20}$$

which is a direct analogue to the unitarity condition for the bulk S -matrix. To obtain the crossing symmetry condition for the boundary scattering, it is necessary to use the ‘cross amplitude’ [5]

$$K^{abc}(\theta) = R_{ba}^c\left(\frac{i\pi}{2} - \theta\right). \tag{21}$$

As shown in [5], this amplitude $K^{abc}(\theta)$ has to satisfy the so-called ‘boundary cross-unitarity condition’

$$K^{abc}(\theta) = \sum_{d \neq a, c} S_{ca}^{db}(2\theta) K^{adc}(-\theta) \tag{22}$$

which is illustrated in figure 7. Finally we have the ‘boundary bootstrap equation’ [11] which describes the scattering of ‘bound-state’ particles with the boundary. In the bulk theory, the kink A_{ab} can appear as a bound-state particle in the two-particle scattering (see figure 2)

$$A_{ac}\left(\theta + \frac{i\pi}{3}\right) A_{cb}\left(\theta - \frac{i\pi}{3}\right) \rightarrow f A_{ab}(\theta) \tag{23}$$

where f is the three-particle coupling in (13). Applying the algebras (8), (18) and (23) to the ‘in-state’ $A_{ac}(\theta + \frac{i\pi}{3}) A_{cb}(\theta - \frac{i\pi}{3}) B_b$, we obtained the bootstrap equation (figure 8)

$$R_{ab}^d(\theta) = \sum_{f \neq c, e} \sum_{e \neq a, d} R_{cb}^f\left(\theta - \frac{i\pi}{3}\right) S_{af}^{ce}(2\theta) R_{ef}^d\left(\theta + \frac{i\pi}{3}\right) \tag{24}$$

for the S_q symmetric Potts model.

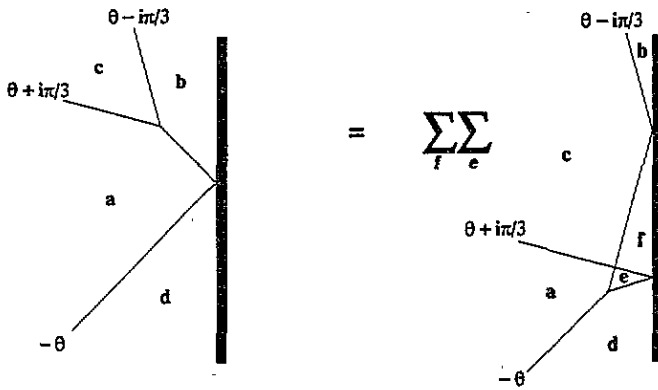


Figure 8. Boundary bootstrap equation where states e and f must satisfy the ‘admissibility condition’ and the boundary condition.

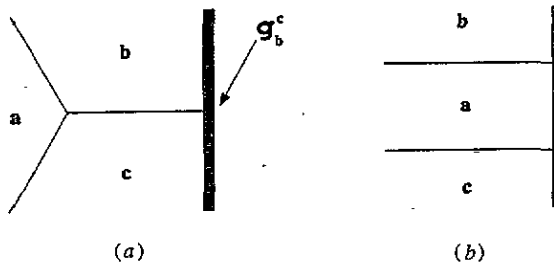


Figure 9. Physical poles of the boundary S-matrix with conditions $a \neq b \neq c$ in (a), and $a \neq b, c$ in (b). Here g_b^c is the boundary coupling constant for the particle A_{cb} .

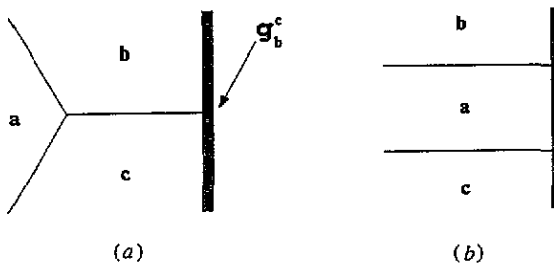


Figure 10. Boundary S-matrix elements for the free boundary condition with $a, b,$ and c all different.

Equations (19) through (24) allow the boundary S-matrix elements $R_{ab}^c(\theta)$ to be determined up to some CDD factors [5]. For boundary conditions which respect the S_q symmetry, we can expect $R_{ab}^c(\theta)$ to have a pole at $\theta = \frac{i\pi}{6}$ (figure 9(a)) with residue

$$R_{ab}^c(\theta) \simeq \frac{i}{2} \frac{f g_b^c}{\theta - \frac{i\pi}{6}} \tag{25}$$

where g_b^c is the amplitude for coupling of the particle A_{cb} to the boundary (figure 10). Furthermore, if $g_a^a, g_b^a \neq 0$, the element $R_{ab}^c(\theta)$ has another pole at $\theta = \frac{i\pi}{2}$ (figure 9(b))

where

$$R_{ab}^c(\theta) \simeq \frac{i g_a^c g_b^c}{2\theta - \frac{i\pi}{2}}. \quad (26)$$

This pole is shown in figure 10. Of course, the presence of the above poles depends on the boundary condition, as we shall see when we consider the two simplest cases: 'free' and 'fixed' boundary conditions. Both cases are conformal boundary conditions (i.e. $\Phi_B(y) = 0$) and we conjecture that they also preserve integrability in the off-critical Potts model.

4. Fixed boundary condition

In this simple boundary condition, the boundary spins are all fixed to one state, say 'a'. The corresponding boundary S -matrix element

$$R_{ba}^a(\theta) = R_f(\theta) \quad (27)$$

satisfies the boundary Yang-Baxter equation (19) automatically. To determine this amplitude, one appeals to the unitarity condition (20)

$$R_f(\theta)R_f(-\theta) = 1 \quad (28)$$

and the crossing symmetry condition (22)

$$K_f(\theta) = [(q-2)S_2(2\theta) + S_3(2\theta)]K_f(-\theta) \quad (29)$$

where

$$K_f(\theta) = R_f\left(\frac{i\pi}{2} - \theta\right) \quad (30)$$

is the crossing amplitude. Since all boundary states are fixed, we do not expect $R_f(\theta)$ to possess any poles in the physical domain $0 \leq \theta \leq \frac{i\pi}{2}$. The solution to (28) and (29) can be factorized as

$$R_f(\theta) = F_0(\theta)F_1\left(\frac{\lambda\theta}{i\pi}\right) \quad (31)$$

where $F_1(X)$ solves

$$F_1(X) = \Pi(\lambda - 2X) \frac{\sin(2\pi(\lambda - X))}{\sin(2\pi(\frac{\lambda}{3} - X))} F_1(\lambda - X). \quad (32)$$

The equation

$$F_1(-X)F_1(X) = 1 \quad (33)$$

and its minimal solution can be written as

$$F_1(X) = \prod_{k=1}^{\infty} \frac{\Sigma_k(X)}{\Sigma_k(-X)} \quad (34a)$$

with

$$\Sigma_k(X) = \frac{\Gamma[(4k-1)\lambda + 2X]\Gamma[(4k-3)\lambda + 2X + 1]\Gamma[(4k-3)\lambda + \frac{\lambda}{3} + 2X]\Gamma[(4k-2)\lambda + \frac{\lambda}{3} - 2X]}{\Gamma[4k\lambda + 2X]\Gamma[4(k-1)\lambda + 2X + 1]\Gamma[(4k-2)\lambda - \frac{\lambda}{3} + 2X + 1]\Gamma[(4k-1)\lambda - \frac{\lambda}{3} - 2X + 1]} \quad (34b)$$

up to some CDD factors.

The factor $F_0(\theta)$ can be obtained from the fixed boundary bootstrap equation (24)

$$R_f(\theta) = [S_1(2\theta) + (q-3)S_0(2\theta)]R_f\left(\theta - \frac{i\pi}{3}\right)R_f\left(\theta + \frac{i\pi}{3}\right) \quad (35)$$

which reduces to

$$F_0(\theta) = -\tan\left(\frac{\pi}{4} + \frac{i\theta}{2}\right)\cot\left(\frac{\pi}{12} + \frac{i\theta}{2}\right)\cot\left(\frac{5\pi}{12} + \frac{i\theta}{2}\right)F_0\left(\theta - \frac{i\pi}{3}\right)F_0\left(\theta + \frac{i\pi}{3}\right) \quad (36)$$

with simple solution

$$F_0(\theta) = -\tan\left(\frac{\pi}{4} + \frac{i\theta}{2}\right). \quad (37)$$

In the Ising limit ($q = 2, \lambda = \frac{3}{4}$), the boundary S-matrix has the form

$$R_f(\theta) = i \tanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right) \quad (38)$$

as obtained in [5] by the field-theoretic method.

In the other interesting limit $q = 3$ ($\lambda = 1$), suppose the boundary is fixed in the state 'A', the solution simplifies as

$$R_f(\theta) = -\frac{\sin(\frac{\pi}{3} + \frac{i\theta}{2})}{\sin(\frac{\pi}{3} - \frac{i\theta}{2})}. \quad (39)$$

5. Free boundary condition

In contrast to the fixed boundary condition, we have the 'free' case where the boundary spin order parameters can be in any one of the q states. The corresponding boundary S-matrix has to respect the S_q symmetry and the algebra (18) simplifies to

$$A_{ba}(\theta)B_a = R_1(\theta)A_{ba}(-\theta)B_a + \sum_{c \neq a,b} R_2(\theta)A_{bc}(-\theta)B_c \quad (40)$$

where the amplitudes $R_1(\theta)$ and $R_2(\theta)$ are shown in figure 10.

The boundary Yang–Baxter equation (19) provides three non-trivial equations:

$$\begin{aligned}
 &R_1 S_3 R_2 S_1 + R_2 S_1 R_1 S_1 + (q - 3) R_2 S_1 R_2 S_1 + (q - 3) R_1 S_2 R_2 S_0 \\
 &\quad + (q - 3) R_2 S_0 R_1 S_0 + (q - 3)(q - 4) R_2 S_0 R_2 S_0 \\
 &= (q - 2) R_2 S_2 R_1 S_2 + (q - 3) R_1 S_0 R_2 S_2 + (q - 3)^2 R_2 S_0 R_2 S_2 \\
 &\quad + R_1 S_1 R_2 S_3 + R_2 S_3 R_1 S_3 + (q - 3) R_2 S_1 R_2 S_3
 \end{aligned} \tag{41a}$$

$$\begin{aligned}
 &R_1 S_1 R_2 S_2 + (q - 3) R_2 S_1 R_2 S_2 + R_2 S_2 R_1 S_3 + (q - 3) R_2 S_0 R_2 S_3 + (q - 3) R_2 S_2 R_1 S_2 \\
 &\quad + R_2 S_3 R_1 S_2 + (q - 3) R_1 S_0 R_2 S_2 + (q - 3)(q - 4) R_2 S_0 R_2 S_2 \\
 &= R_1 S_2 R_2 S_1 + (q - 3) R_1 S_2 R_2 S_0 + (q - 3) R_2 S_0 R_2 S_1 + (q - 3)^2 R_2 S_0 R_2 S_0
 \end{aligned} \tag{41b}$$

$$\begin{aligned}
 &R_1 S_3 R_2 S_0 + R_1 S_2 R_2 S_1 + (q - 4) R_1 S_2 R_2 S_0 + R_2 S_1 R_1 S_0 + R_2 S_0 R_1 S_1 \\
 &\quad + (q - 4) R_2 S_0 R_1 S_0 + (q - 3) R_2 S_1 R_2 S_0 + (q - 4) R_2 S_0 R_2 S_1 + (q - 4)^2 R_2 S_0 R_2 S_0 \\
 &= R_1 S_1 R_2 S_2 + R_1 S_0 R_2 S_3 + (q - 4) R_1 S_0 R_2 S_2 + R_2 S_3 R_1 S_2 + R_2 S_2 R_1 S_3 \\
 &\quad + (q - 3) R_2 S_2 R_1 S_2 + (q - 3) R_2 S_1 R_2 S_2 + (q - 4) R_2 S_0 R_2 S_3 \\
 &\quad + [(q - 3) + (q - 4)^2] R_2 S_0 R_2 S_2
 \end{aligned} \tag{41c}$$

where the argument in each term has the form $R_i(\theta_1)S_j(\theta_1 + \theta_2)R_k(\theta_2)S_l(\theta_2 - \theta_1)$.

Equation (41) can be solved for the ratio $R_1(\theta_1)/R_2(\theta_1)$ when we take the limit $\theta_2 \rightarrow \frac{i\pi}{2}$ and noting that both $R_1(\theta_2)$ and $R_2(\theta_2)$ have a simple pole in this limit with the same residue

$$\text{Res}_{\theta_2=\frac{i\pi}{2}} R_1(\theta_2) = \text{Res}_{\theta_2=\frac{i\pi}{2}} R_2(\theta_2) = \frac{i}{2} g^2 \tag{42}$$

for some boundary coupling g .

The solution to (41) can then be written as

$$R_1(\theta) = (q - 3) \frac{\sinh \lambda(\frac{i\pi}{3} + 2\theta)}{\sinh \lambda(i\pi - 2\theta)} P\left(\frac{\lambda\theta}{i\pi}\right) \tag{43a}$$

$$R_2(\theta) = \frac{\sin \frac{2\pi\lambda}{3}}{\sin \frac{\pi\lambda}{3}} \frac{\sinh 2\lambda\theta}{\sinh \lambda(i\pi - 2\theta)} \frac{\sinh \lambda(\frac{i\pi}{3} + 2\theta)}{\sinh \lambda(\frac{i\pi}{3} - 2\theta)} P\left(\frac{\lambda\theta}{i\pi}\right) \tag{43b}$$

where we use the fact that $R_2(\theta)$ has a simple pole at $\theta = \frac{i\pi}{6}$, which is absent in $R_1(\theta)$. We do not expect $R_1(\theta)$ and $R_2(\theta)$ to have any other poles in the physical domain ($0 \leq \theta \leq \frac{i\pi}{2}$). The normalization factor $P(\frac{\lambda\theta}{i\pi})$ is constrained by the unitarity conditions (20) and (21), which reduce to

$$P(\theta)P(-\theta) = 1 \tag{44a}$$

and

$$P\left(\frac{i\pi}{2} - \theta\right) = -\Pi\left(\frac{\lambda\theta}{i\pi}\right) \frac{\sinh \lambda(i\pi + \theta)}{\sinh \lambda(\frac{i\pi}{3} - \theta)} \frac{\sinh \lambda(\frac{4i\pi}{3} + \theta)}{\sinh \lambda(\frac{4i\pi}{3} - \theta)} P\left(\frac{i\pi}{2} + \theta\right) \tag{44b}$$

respectively. The minimal solutions can be written as

$$P(X) = \prod_{k=1}^{\infty} \frac{\Omega_k(X)}{\Omega_k(-X)} \tag{45a}$$

where

$$\Omega_k(X) = \frac{\Gamma[(4k-1)\lambda + 2X]\Gamma[(4k-3)\lambda + 2X + 1]\Gamma[4k\lambda + \frac{\lambda}{3} - 2X]\Gamma[4(k-1)\lambda - \frac{\lambda}{3} - 2X + 1]}{\Gamma[4k\lambda + 2X]\Gamma[4(k-1)\lambda + 2X + 1]\Gamma[(4k-3)\lambda + \frac{\lambda}{3} - 2X]\Gamma[(4k-1)\lambda - \frac{\lambda}{3} - 2X + 1]} \quad (45b)$$

up to some CDD factors. The sign in (45) can be justified by the boundary bootstrap equation (24).

In the Ising limit ($\lambda \rightarrow \frac{3}{4}$), the boundary S -matrices element simplifies to

$$R_1(\theta) = -\cot\left(\frac{\pi}{4} + \frac{i\theta}{2}\right) \quad (46)$$

in agreement with [5]. For the $q = 3$ Potts model ($\lambda \rightarrow 1$), the two scattering amplitudes have the form

$$R_1(\theta) = 0$$

$$R_2(\theta) = \frac{\sin(\frac{\pi}{12} - \frac{i\theta}{2}) \sin(\frac{\pi}{4} - \frac{i\theta}{2})}{\sin(\frac{\pi}{12} + \frac{i\theta}{2}) \sin(\frac{\pi}{4} + \frac{i\theta}{2})}. \quad (47)$$

Finally, a simple computation gives us

$$g(\lambda) = -\sqrt{\frac{(3-q)}{\lambda} \sin\left(\frac{4\pi}{3}\lambda\right)} \exp\left\{\int_0^\infty \frac{dt}{t} \frac{e^{-t-4\lambda t}}{2(1-e^{-t})(1+e^{-2\lambda t})} \times [(1-e^{-\lambda t})(1-e^{3\lambda t-t}) + e^{\frac{\lambda}{3}-t}(1-e^{-2\lambda t+\frac{\lambda}{3}+t})(1-e^{3\lambda t})]\right\} \quad (48)$$

for the boundary coupling constant.

6. Conclusion

In this study, the q -Potts model boundary S -matrices for free and fixed boundary conditions were derived. It would be interesting to apply the techniques of the thermodynamics Bethe ansatz to study these S -matrices. In particular, one can investigate the renormalization-group flow between various boundary conditions [5, 13]. This work is in progress. It may also be possible to use these boundary S -matrices to compute the crossing probabilities for the percolation problem ($q = 1$) in a finite region [12]. Recently, quantum field theories with boundary were successfully applied to study the s -wave scattering of electrons off an impurity (for a review see [14]). It would be worth determining what scattering phenomena exhibit the boundary S -matrix structure of the q -Potts model. Finally we would like to add that boundary S -matrices can in fact be derived from the lattice model itself [15, 16], and it seems to be an interesting exercise to carry out such a construction for the q -Potts model.

Acknowledgments

LC was supported by an Eleanor Sophia Wood Travelling Scholarship (Sydney University) and would like to thank Professor A B Zamolodchikov for many useful discussions.

References

- [1] Itzykson E, Saleur H and Zuber J B (ed) 1988 *Conformal Invariance and Applications to Statistical Mechanics* (Singapore: World Scientific)
- [2] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys. B* **241** 33
- [3] Zamolodchikov A B 1989 *Adv. Stud. Pure Math.* **19** 641
- [4] Smirnov F 1992 *Formfactors in Completely Integrable Models of QFT* (Adv. Series Math. Phys. 14) (Singapore: World Scientific)
- [5] Ghoshal S and Zamolodchikov A 1994 *Int. J. Mod. Phys. A* **9** 3841
- [6] Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106
- [7] Dotsenko V I S and Fateev V A 1984 *Nucl. Phys. B* **240** 312
- [8] Chim L and Zamolodchikov A 1992 *Int. J. Mod. Phys. A* **7** 5317
- [9] Cardy J 1989 *Nucl. Phys. B* **324** 581
- [10] Cherednik I 1984 *Theor. Math. Phys.* **61** 35 977
- [11] Fring A and Koberle R 1994 *Nucl. Phys. B* **421** 159
- [12] Cardy J 1992 *J. Phys. A: Math. Gen.* **25** L201
- [13] Fendley P and Saleur H 1994 Exact theory of polymer adsorption in analogy with the Kondo problem
Preprint USC-94-006, CM/9403095
- [14] Ludwig A W W 1994 *Int. J. Mod. Phys. B* **8** 347
- [15] Fendley P and Saleur H 1994 *Nucl. Phys. B* **428** 681.
- [16] Grisaru M T, Mezincescu L and Nepomechie R I 1995 *J. Phys. A: Math. Gen.* **28** 1027